arXiv:0709.0011v2 [math.OA] 14 Oct 2007

FREENESS WITH AMALGAMATION, LIMIT THEOREMS AND S-TRANSFORM IN NON-COMMUTATIVE PROBABILITY SPACES OF TYPE B

MIHAI POPA

ABSTRACT. The present material addresses several problems left open in the Trans. AMS paper " Non-crossing cumulants of type B" of P. Biane, F. Goodman and A. Nica. The main result is that a type B non-commutative probability space can be studied in the framework of freeness with amalgamation. This view allows easy ways of constructing a version of the S-transform as well as proving analogue results to Central Limit Theorem and Poisson Limit Theorem.

1. INTRODUCTION

The present material addresses several problems left open in the paper "Non-crossing cumulants of type B" of P. Biane, F. Goodman and A. Nica (reference [4]).

The type A, B, C and D root systems determine correspondent lattices of non-crossing partitions (see [8], [2]). The type A_{n+1} corresponds to the lattice of non-crossing partitions on the ordered set $[n] = 1 < \cdots < n$; the types B_n and C_n determine the same lattice of non-crossing partitions on $[\overline{n}] = 1 < \cdots < n < -1 < \cdots < -n$, namely the partitions with the property that if V is a block, then $-V$ (the set containing the opposites of the elements from V) is also a block; the type D corresponds to a lattice of the symmetric non-crossing partitions with the property that if there exists a symmetric block, then it has more than 2 elements and contains $-n$ and n. (see again [8], [2], [3]).

The lattices of type A and type B non-crossing partitions are self-dual with respect to the Kreweras complementary. In the type A case, the lattice structure was known to be connected the combinatorics of Free Probability Theory (see [7]). For the type B case, the properties of the lattice allow also a construction, described in [4], of some associated non-commutative probability spaces, with a similar apparatus as in the type A case (such as R-transform and boxed convolution). The paper [4] leaves open some questions on these objects: possible connections to other types of independence, limit theorems, S-transform. The main observation of the present material is that a type B non-commutative probability space can be studied in the framework of freeness with amalgamation, that gives fast answers to the rest of the problems.

The material is organized as follows: second section reviews some results from [4]; third section presents the connection with freeness with amalgamation; forth section is briefing the construction of the S-transform for the type B non-commutative probability spaces, utilizing the commutativity of the matrix algebra C; fifth and,

respectively, sixth section are presenting limit results: analogues of central limit theorem, respectively Poisson limit theorem.

2. PRELIMINARY RESULTS

Definition 2.1. A non-commutative probability space of type B is a system $(\mathcal{A}, \varphi, \mathcal{X}, f, \Phi)$, where:

- (i) $(\mathcal{A}, \varphi)$ is a non-commutative probability space (of type A), i.e. $\mathcal{A}$ is a complex unital algebra and $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ is a linear functional such that $\varphi(1) = 1$.
- (ii) $\mathcal{X}$ is a complex vector space and $f : \mathcal{X} \longrightarrow \mathbb{C}$ is a linear functional.
- (iii) $\Phi : \mathcal{A} \times \mathcal{X} \times \mathcal{A} \longrightarrow \mathcal{A}$ is a two-sided action of $\mathcal{A}$ on ν (when there is no confusion, it will be written "$a\xi b$ *instead of* $\Phi(a\xi b)$, *for* $a, b \in \mathcal{A}$ *and* $\xi \in \nu$)

On the vector space $\mathcal{A}\times\mathcal{X}$ it was defined a structure of unital algebra considering the multiplication:

$$(a, \xi) \cdot (b, \eta) = (ab, a\eta + \xi b),\ \ a, b \in \mathcal{A},\ \ \xi, \eta \in \mathcal{X}$$

The above algebra structure can be obtained when $(a, \xi) \in \mathcal{A} \times \mathcal{X}$ is identifies with a 2×2 matrix,

$$(a, \xi) \leftrightarrow \begin{bmatrix} a & \xi \\ 0 & a \end{bmatrix}.$$

We will consider also the commutative unital algebra $\mathcal{C}$ by similarly endowing the vector space $\mathbb{C} \times \mathbb{C}$ with the multiplication:

$$(x, t) \cdot (y, s) = (xy, xs + ty),$$

i.e. using the identification

$$\mathcal{C} \ni (x, t) \leftrightarrow \begin{bmatrix} x & t \\ 0 & x \end{bmatrix} \in M_2(\mathbb{C}).$$

Definition 2.2. Let $(\mathcal{A}, \varphi, \mathcal{X}, f, \Phi)$ be a non-commutative probability space of type B. The non-crossing cumulant functionals of type B are the families of multilinear functionals $(\kappa_n : (\mathcal{A} \times \mathcal{X})^n \longrightarrow \mathcal{C})_{n=1}^{\infty}$ defined by the following equations: for every $n \geq 1$ and every $a_1, \dots, a_n \in \mathcal{A}, \xi_1, \dots, \xi_n \in \mathcal{X}$, we have that:

$$\sum_{\gamma\in NC^{(A)}(n)} \prod_{B\in\gamma} \kappa_{card(B)} \left((a_1, \xi_1) \cdots (a_n, \xi_n)|B\right) = E\left((a_1, \xi_1) \cdots (a_n, \xi_n)\right) \tag{1}$$

where the product on the left-hand side is considered with respect to the multiplication on $\mathcal{C}$ and the product $(a_1, \xi_1) \cdots (a_n, \xi_n)$ on the right-hand side is considered with respect to the multiplication on $\mathcal{A} \times \mathcal{X}$ defined above.

Note that the first component of $\kappa_m \left((a_1, \xi_1) \cdots (a_n, \xi_m)\right)$ equals the non-crossing cumulant $k_m(a_1, \dots, a_m)$.

We will also use the notation $\kappa_n(a, \xi)$ for $\kappa_n \left((a, \xi) \cdots (a, \xi)\right)$ and M_n for $E\left((a, \xi)^n\right)$.

Definition 2.3. Let $\mathcal{A}_1, \dots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$ and let $\mathcal{X}_1, \dots, \mathcal{X}_k$ be linear subspaces of $\mathcal{X}$ such that each $\mathcal{X}_j$ is invariant under the action of $\mathcal{A}_j$. We say that $(\mathcal{A}_1, \mathcal{X}_1), \dots (\mathcal{A}_k, \mathcal{X}_k)$ are free independent if

$$\kappa_n \left((a_1, \xi_1), \dots, (a_n, \xi_n)\right) = 0$$

whenever $a_l \in \mathcal{A}_{i_l}, \xi_l \in \mathcal{X}_{i_l}$ $(l = 1, \dots, n)$ are such that there exist $1 \leq s < t \leq n$ with $i_s \neq i_t$.

For $(a, \xi) \in \mathcal{A} \times \mathcal{X}$ we consider the moment and cumulat or R-transform, series:

$$\begin{aligned} M(a, \xi) &= \sum_{n=1}^{\infty} \left(E\left((a, \xi)^n\right)\right) z^n \\ R(a, \xi) &= \sum_{n=1}^{\infty} \kappa_n(a, \xi) z^n \end{aligned}$$

Definition 2.4. Let $\Theta^{(B)}$ be the set of power series of the form:

$$f(z) = \sum_{n=1}^{\infty} (\alpha'_n, \alpha''_n) z^n,$$

where α'_n, α''_n are complex numbers. For $p \in NC^{(A)}(n)$ and $f \in \Theta^{(B)}$, consider

$$Cf_p(f) = \prod_{B \in p} (\alpha'_{|B|}, \alpha''_{|B|})$$

(the right-hand side product is in $\mathcal{C}$.) On $\Theta^{(B)}$ we define the binary operation $\boxtimes$ by:

$$\begin{aligned} f \boxtimes g &= \sum_{n=1}^{\infty} (\gamma'_n, \gamma''_n) z^n \ \ \textit{where} \\ (\gamma'_n, \gamma''_n) &= \sum_{p \in NC^{(A)}(n)} Cf_p(f) Cf_{Kr(p)}(g) \end{aligned}$$

Theorem 2.5. The moment series M and R-transform R of (a, ξ) are related by the formula

$$M = R \boxtimes \zeta'$$

where $\zeta' \in \Theta^{(B)}$ is the series $\sum_{n=1}^{\infty} (1, 0) z^n$.

Remark 2.6. We denote by $k'_{n,p}$ or, for simplicity, by k'_n, the multilinear functional from $\mathcal{A}^{p-1} \times \mathcal{X} \times \mathcal{A}^{n-p}$ to $\mathbb{C}$ which is defined by the same formula as for the (type A) free cumulants $k^n : \mathcal{A}^n \longrightarrow \mathbb{C}$, but where the pth argument is a vector from $\mathcal{X}$ and φ is replaced by f in all the appropriate places. The connexion between the type B cumulants κ_n and the functionals k_n, k'_n is given by:
(2)

$$\kappa_n((a_1, \xi_1), \ldots, (a_n, \xi_n)) = \left(k_n(a_1, \ldots, a_n), \sum_{p=1}^{n} k'_n(a_1, \ldots, a_{p-1}, \xi_p, a_{p+1}, \ldots, a_n) \right)$$

Theorem 2.7. If $(\mathcal{A}_1, \mathcal{X}_1), (\mathcal{A}_2, \mathcal{X}_2)$ are free independent, $(a_1, \xi_1) \in (\mathcal{A}_1, \mathcal{X}_1), (a_2, \xi_2) \in (\mathcal{A}_2, \mathcal{X}_2)$, and R_1, respectively R_2 denote the R-transforms of (a_1, ξ_1) and (a_2, ξ_2), then:

(i) the R-transform of $(a_1, \xi_1) + (a_2, \xi_2)$ is $R_1 + R_2$.
(2) the R-transform of $(a_1, \xi_1) \cdot (a_2, \xi_2)$ is $R_1 \boxtimes R_2$.

3. CONNEXION TO "FREENESS WITH AMALGAMATION"

As shown in [4], Section 6.3, Remark 3, the definitions of the type B cumulants are close to those from the framework of the "operator-valued cumulats", yet some detailes are different - mainly the map E is not a conditional expectation and $\mathcal{A} \times \mathcal{X}$ is not a bimodule over C. Following a suggestion of Dimitri Shlyakhtenko,

the construction of the type B probability spaces can still be modified in order to overcome these points.

Let $\mathfrak{C} = \mathcal{X} \oplus \mathcal{A}$. On $\mathcal{A} \times \mathfrak{C}$ we have a C-bimodule structure given by:

$$(x,t)(a,\xi+b) = (a,\xi+b)(x,t) = (ax, at + (\xi+b)x)$$

for any $x, t \in \mathbb{C}, a, b \in \mathcal{A}, \xi \in \mathcal{X}$. Since $\mathcal{A}$ is unital, C is a subspace of $\mathfrak{C}$.

The map E extends to $\mathfrak{C}$ via:

$$\widetilde{E}(a,\xi+b) = (\varphi(a), f(\xi) + \varphi(b))$$

The extension becomes a conditional expectation, since:

$$\begin{aligned}
\widetilde{E}\left((x,t)(a,\xi+b)\right) &= \widetilde{E}(ax, at + (\xi+b)x) \\
&= (\varphi(ax), \varphi(ta) + f(\xi x) + \varphi(bx)) \\
&= (x\varphi(a), t\varphi(a) + xf(\xi) + x\varphi(b)) \\
&= (x,t)\left(\varphi(a), f(\xi) + \varphi(b)\right) \\
&= (x,t)\widetilde{E}(a,\xi+b)
\end{aligned}$$

The equation 1 can naturally be extended in the framework of $\mathfrak{C}$ and $\widetilde{E}$, framework that reduces the construction to freeness with amalgamation, namely defining the cumulants $\widetilde{\kappa}$ by the equation:

$$\sum_{\gamma \in NC^{(A)}(n)} \prod_{B \in \gamma} \widetilde{\kappa}_{card(B)}\left((a_1,\xi_1)\cdots(a_n,\xi_n)|B\right) = \widetilde{E}\left((a_1,\xi_1)\cdots(a_n,\xi_n)\right) \tag{3}$$

If $m : \mathcal{A} \times \mathcal{A} \ni (a,b) \mapsto m(a,b) = ab \in \mathcal{A}$ is the multiplication in $\mathcal{A}$, note that $(\mathcal{A}, \varphi, \mathcal{X} \oplus \mathcal{A}, f \oplus \varphi, \Phi \oplus m)$ is also a type B noncommutative probability space, therefore Remark 2.6 (i.e. Theorem 6.4 from [4]) gives the components of $\widetilde{\kappa}$:

$$\begin{aligned}
\widetilde{\kappa}_n((a_1,\xi_1+b_1),\ldots,(a_n,\xi_n+b_n)) &= \\
&\left(k_n(a_1,\ldots,a_n), \sum_{p=1}^{n} k'_n(a_1,\ldots,a_{p-1},\xi_p+b_p,a_{p+1},\ldots,a_n)\right)
\end{aligned}$$

4. The S-transform

Utilizing the commutativity of the algebra C, the construction of the S-transform is essentially a verbatim reproduction of the type A situation.

We will denote

$$\mathcal{G} = \{\sum_{n=1}^{\infty} \alpha_n z^n, \alpha_n \in C\}$$

the set of formal series without constant term with coefficients in C, and

$$\mathcal{G}^{\langle -1 \rangle} = \{\sum_{n=1}^{\infty} \alpha_n z^n, \alpha_n \in C, \alpha_1 = \text{invertible}\}$$

the set of all invertible series (with respect to substitutional composition) with coefficients in C (see [1]).

Definition 4.1. Let $(a,\xi) \in \mathcal{A} \times \mathcal{X}$ such that $\varphi(a) \neq 0$, that is $(\varphi(a), f(\xi))$ is invertible in C. If $R_{(a,\xi)}(z)$ is the R-transform series of (a,ξ), then the S-*transform* of (a,ξ) is the series defined by

$$S_{(a,\xi)}(z) = \frac{1}{z} R^{\langle -1 \rangle}_{(a,\xi)}(z)$$

Theorem 4.2. If $\mathcal{A}_1, \mathcal{X}_1), (\mathcal{A}_2, \mathcal{X}_2) \subset (\mathcal{A}, \mathcal{X})$ are free independent and $(x_j, \xi_j) \in (\mathcal{A}_j, \mathcal{X}_j), j = 1,2$ are such that $\varphi(x_j) \neq 0$, then:

$$S_{(a_1,\xi_1)(x_2,\xi_2)}(z) = S_{(a_1,\xi_1)}(z) S_{(a_2,\xi_2)}(z)$$

Proof. The proof presented in [7], for the type A case, works also for the freeness with amalgamation over a commutative algebra. Yet, for the convenience of the reader, we will outline the main steps.

Since, for $(a_1,\xi_1), (a_2,\xi_2)$ free, $R_{(a_1,\xi_1)\cdot(a_2,\xi_2)} = R_{(a_1,\xi_1)} \boxed{\star} R_{(a_2,\xi_2)}$, it suffices to prove that the mapping

$$\mathcal{F} : \mathcal{G}^{\langle -1 \rangle} \ni f \mapsto \frac{1}{z} f^{\langle -1 \rangle} \in \mathcal{G}$$

has the property

$$\mathcal{F}\left(f \boxed{\star} g\right) = \mathcal{F}(f)\mathcal{F}(g). \tag{4}$$

Indeed, (4) is equivalent to

$$z\left(f \boxed{\star} g\right) = f^{\langle -1 \rangle}\left(f \boxed{\star} g\right) \cdot g^{\langle -1 \rangle}\left(g \boxed{\star} f\right) \tag{5}$$

For $\sigma \in NC(n)$ and $h = \sum_{n \geq 1} h_n z^n$, we define

$$Cf_\sigma(h) = \prod_{B \in \sigma} h_{card(B)} \in C.$$

Also, for $f, g \in \mathcal{G}$, we denote

$$\left(f \check{\boxed{\star}} g\right)(z) = \sum_{n \geq 1} \lambda_n z^n$$

where ($K(\sigma)$ is the Kreweras complementary of σ)

$$\lambda_n = \sum_{\substack{\sigma \in NC(n) \\ (1) \text{ block in } \sigma}} Cf_\sigma(f) \cdot Cf_{K(\sigma)}(g)$$

For $f = \sum_{n \geq 1} \alpha_n z^n \in \mathcal{G}^{\langle -1 \rangle}$ we have that:

$$f^{\langle -1 \rangle} \circ \left(f \boxed{\star} g\right) = \alpha_1^{-1}\left(f \check{\boxed{\star}} g\right)$$

since, with the above notations, the coefficient of z^m in the right hand side is

$$\sum_{n \geq 1} \sum_{\substack{i_1, \dots, i_n \geq 1 \\ i_1 + \dots + i_n = m}} \alpha_n \alpha_1^{-n} \lambda_{i_1} \cdots \lambda_{i_n}$$

while the coefficient of z^m in the left-hand side is

$$\sum_{n \geq 1} \sum_{1 = b_1 < \dots b_n \leq m} \sum_{\substack{\pi \in NC(m) \\ (b_1, \dots, b_n) \in \pi}} Cf_\pi(f) \cdot Cf_{K(\pi)}(g)$$

and the equality follows setting $\pi_k = \pi|\{b_k, \dots, b_{k+1}-1\}$ (notationally $b_{n+1} = m$) and remarking that $K(\pi)$ is the juxtaposition of $K(\pi_1), \dots, K(\pi_n)$.

It follows that , if $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are respectively the coefficients of f and g,(5) is equivalent to

$$\left(f \check{\boxed{\star}} g\right)(z) \cdot \left(f \check{\boxed{\star}} g\right)(z) = \alpha_1\beta_1 z \cdot \left(f \boxed{\star} g\right)(z)$$

The coefficient of z^{m+1} on the left-hand side is

$$\sum_{n=1}^{m} \sum_{\substack{\pi \in NC(n) \\ (1) \in \pi}} \sum_{\substack{\rho \in NC(m+n-1) \\ (1) \in \rho}} Cf_\pi(f) \cdot Cf_{K(\pi)}(g) \cdot Cf_\rho(g) \cdot Cf_{K(\rho)}(f)$$

while the coefficient of z^{m+1} on the right-hand side is

$$\sum_{\sigma \in NC(m)} \alpha_1\beta_1 \cdot Cf_\sigma(f) \cdot Cf_{K(\sigma)}(g).$$

As shown in [7], the conclusion follows from the bijection between the index sets of the above sums. More precisely, if $1 \leq n \leq m$, to the pair consisting on $\pi \in NC(n)$ and $\rho \in NC(m+1-n)$ both contain the block (1), we associate the partition from $NC(n+m-1)$ obtained by juxtaposing $\pi \setminus (1)$ and $K(\rho)$. $\square$

5. CENTRAL LIMIT THEOREM

Theorem 5.1. Let $\{(\mathcal{A}_k, \mathcal{X}_k)\}_{k\geq 1} \subset (\mathcal{A}, \mathcal{X})$ be type B free independent and $(x_k, \xi_k) \in (\mathcal{A}_k, \mathcal{X}_k)$ identically distributed such that $\varphi(x_k) = f(\xi_k) = 0$ and $\varphi(x_k^2) = f(\xi_k^2) = 1$. The limit distribution moments of

$$\frac{(a_1, \xi_1) + \cdots + (a_N, \xi_N)}{\sqrt{N}}$$

are $\{m_n, \mathfrak{m}_n\}_n$, where $\{m_n\}_n$ are the moments of the semicircular distribution and

$$\mathfrak{m}_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{2k}{k+1} & \text{if } n = 2k \text{ is even.} \end{cases}$$

Proof. Note $S_N = \frac{(a_1,\xi_1)+\cdots+(a_N,\xi_N)}{\sqrt{N}}$ and $R_N = R(S_N)$. Theorem 2.7 implies

$$\lim_{N\to\infty} R_N = (1,1)z^2$$

The first component of the limit distribution is the Voiculescu's semicircular distribution. To compute the second component of the moments, we will use the equation (1), which becomes:

$$E\left((a_1,\xi_1)^n\right) = \sum_{\gamma \in NC_2^{(A)}(n)} \kappa_2\left((a_1,\xi_1)\right)^{\frac{n}{2}}$$

It follows that all the odd moments are zero, and, since in C, $(a,b)^n = (a^n, na^{n-1}b)$, the even moments are given by:

$$\begin{aligned}\mathfrak{m}_{2n} &= nC_n, \text{ where } C_n \text{ stands for the } n\text{-th Catalan number}\\ &= n\frac{1}{n+1}\binom{2n}{n}\\ &= \binom{2n}{n+1}.\end{aligned}$$

□

Remark 5.2. The second components of the above limit moments are not the moments of positive Borel measure on $\mathbb{R}$. Yet, they are connected to the moments of another remarkable distribution appearing in non-commutative probability - the central limit distribution for monotonic independence.

For variables that are monotonically independent (see [5], [6]), the limit moments in the Central Limit Theorem are given by the "arsine law", i.e. the n-th moment μ_n is given by

$$\mu_n = \begin{cases} 0 & \textit{if } n \textit{ is odd} \\ \binom{2k}{k} = (k+1)C_k & \textit{if } n = 2k \textit{ is even.} \end{cases}$$

Hence $\mu_n = m_n + \mathfrak{m}_n$, which implies the following:

Corollary 5.3. On $\mathcal{A} \oplus \mathcal{X}$ consider the algebra structure given by:

$$(a+\xi)(b+\eta) = ab + \xi b + a\eta$$

and $\Psi : \mathcal{A} \oplus \mathcal{X} \ni a + \xi \mapsto \varphi(a) + f(\xi) \in \mathbb{C}$.

Let $(a_j, \xi_j)_j = 1^\infty$ be a family from $\mathcal{A} \oplus \mathcal{X}$ such that $\varphi(a_j) = f(\xi_j) = 0$ and (a_j, ξ_j) are type B free in $(\mathcal{A}, \varphi, \mathcal{X}, f, \Phi)$.

Then the limit in distribution of

$$\frac{a_1 + \xi_1 + \dots a_N + \xi_N}{\sqrt{N}}$$

is the "arcsine law".

6. Poisson limit theorem

We will consider an analogue of the classical Bernoulli distribution in a type B probability space.

Let $A = (\alpha_1, \alpha_2) \in \mathbb{R}^2 \subset \mathcal{C}$. We call an element $(a, \xi) \in \mathcal{A} \times \mathcal{X}$ type B Bernoulli with rate Λ and jump size A if

$$E\left((a,\xi)^n\right) = \Lambda A^n$$

for some $\Lambda = (\lambda_1, \lambda_2) \in \mathcal{C}$

Theorem 6.1. Let $\Lambda \in \mathcal{C}$ and $A \in \mathbb{R}^2$. Then the limit distribution for $N \to \infty$ of the sum of N free independent type B Bernoulli variables with rate $\frac{\Lambda}{N}$ and jump size A has cumulants which are given by $\kappa_n = \Lambda A^n$.

Proof. We will introduce first several new notations in order to simplify the writing. β_N will stand for a type B Bernoulli variable with rate $\frac{\Lambda}{N}$ and s_N for a sum of N such free independent variables. μ will denote the Moebius function of the lattice $NC(n)$ and, for $\pi \in NC(n)$ and $\beta \in \mathcal{A} \times \mathcal{X}$, we will use the notation

$$M_\pi(\beta) = \prod_{B = \text{block of} \pi} M_{card(B)}(\beta)$$

where $M_n(\beta) = E(\beta^n)$ is the n-th moment of β.

With the above notations, equation (1) gives

$$\begin{aligned} \kappa_n(\beta_N) &= \sum_{\pi \in NC(n)} M_\pi(\beta_N)\mu(\pi, 1_n) \\ &= \frac{\Lambda}{N}A^n + \sum_{\substack{\pi \in NC(n) \\ 1_n \neq \pi}} M_\pi(\beta_N)\mu(\pi, 1_n) \\ &= \frac{\Lambda}{N}A^n + O\left(\frac{1}{N^2}\right) \end{aligned}$$

Therefore

$$\lim_{N \to \infty} \kappa_n(s_N) = \lim_{N \to \infty} N\kappa_n(\beta_N) = \Lambda A^n.$$

□

Like in the type A case, we have the following:

Consequence 6.2. The square of a type B random variable (a, ξ) with distribution given by the central limit theorem such that $E\left((a, \xi)^2\right) = \sigma \in \mathcal{C}$ is a type B free Poisson element of rate σ and jump size $(1, 0)$.

Remark 6.3. The first component of the moments of a type B free Poisson variable coincides to the type A case, therefore are given by a probability measure on $\mathbb{R}$. In general, the second component of the moments of a type B free Poisson random variable are not the moments of a real measure.

The first part of the assertion is clear. For the second part, we will consider the particular case when $\lambda_2 = 0$ and $\lambda_1 = \lambda$ is close to 0 and $\alpha_1 = \alpha_2 = \alpha$. It follows that

$$\kappa_n = \Lambda A^n = ((\lambda, 0)(\alpha^n, n\alpha^n)).$$

Since equation (1) implies

$$\begin{aligned} M_2 &= \kappa_2 + \kappa_1^2 = (\lambda + \lambda^2)A \\ M_3 &= \kappa_3 + 3\kappa_1\kappa_2 + \kappa_1^3 \\ &= (\lambda + 3\lambda^2 + \lambda^3)A^3 \\ M_4 &= \kappa_4 + 4\kappa_1\kappa_3 + 2\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4 \\ &= (\lambda + 6\lambda^2 + 6\lambda^3 + \lambda^4)A^4 \end{aligned}$$

we have that the second components are given by:

$$\begin{aligned} \mathfrak{m}_2 &= 2(\lambda + \lambda^2)\alpha^2 \\ \mathfrak{m}_3 &= 3(\lambda + 3\lambda^2 + \lambda^3)\alpha^3 \\ \mathfrak{m}_4 &= 4(\lambda + 6\lambda^2 + 6\lambda^3 + \lambda^4)\alpha^4 \end{aligned}$$

A necessary condition for $\{\mathfrak{m}_k\}_{k\geq 1}$ to be the moments of a measure on $\mathbb{R}$ (see [9], [7]) is that

$$\mathfrak{m}_2\mathfrak{m}_4 \geq \mathfrak{m}_3^2$$

It amounts to

$$8(\lambda+\lambda^2)(\lambda+6\lambda^2+6\lambda^3+\lambda^4)\alpha^6 \geq 9(\lambda+3\lambda^2+\lambda^3)^2\alpha^6$$

that is

$$\begin{aligned} 8(1+\lambda)(1+6\lambda+6\lambda^2+\lambda^3) &\geq 9(1+3\lambda+\lambda)^2 \\ 8+O(\lambda) &\geq 9+O(\lambda) \end{aligned}$$

which, for λ small enough, does not hold true.

References

[1] M. Anshelevich, E. G. Effros, M. Popa. Zimmermann type cancellation in the free Faà di Bruno algebra, J. Funct. Anal. 237 (2006), 76-104.

[2] C. A. Athanasiadis. On Noncrossing and Nonnesting Partitions for Classical Reflection Groups, Electronic Journal of Combinatorics, Vol 5 (1998)

[3] C. A. Athanasiadis, V. Reiner. Noncrossing partitions for the group Dn, SIAM J. Discrete Math. 18 (2004), 397-417

[4] P. Biane, F. Goodman, A. Nica. Non-crossing cumulants of type B, Trans. Amer. Math. Soc. 355 (2003), 2263-2303

[5] N. Muraki. Monotonic convolution and monotonic Levy-Hinčin formula, preprint, 2000

[6] N. Muraki. Monotonic independence, montonic central limit theorem and montonic law of small numbers, Infin. Dimen. Anal. Quantum Prob. Relat. Top. 4 (2001) No. 1, 39-58

[7] A. Nica, R. Speicher. Lectures on the Combinatorics of Free Probability London Mathematical Society Lecture Note Series:335, Combridge University Press 2006

[8] V. Reiner. Non-crossing partitions for classical reflection groups. Discrete Math. 177 (1997), 195-222

[9] B. Simon. The Classical Moment Problem as a Self-Adjoint Finite Difference Operator, Advances in Mathematics, 137 (1998), 82-203

[10] R. Speicher. Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, Memoir of the AMS, vol 132, No 627 (1998)

Indiana University at Bloomington, Department of Mathematics, Rawles Hall, 931 E 3rd St, Bloomington, IN 47405

E-mail address: mipopa@indiana.edu

Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764 RO-014700 Bucharest, Romania